SYMBOLIC DYNAMICS AND THE BIFURCATIONS OF A REAL RATIONAL MAP

João Cabral, *Ponta Delgada, Portugal*

2000 MSC: 37E05

Abstract: The importance of symbolic systems is that they give us the possibility of simplifying some dynamical systems. Our expectations is that our work can help to understand better the behaviour, under iteration, of the Real Rational Map, with one parameter

$$f_\lambda(x) = 1 - \frac{2\lambda}{x^2 + \lambda - 4}.$$

Bifurcation theory is a powerful tool, mainly used in Engenhary problems, since that the control of one system's parameter allow us to avoid an undesirable change in the system , or can promote some characteristic of the system, in order to simplify some complicated behaviour to a simpler one.

1 Introduction

Historically, the matter that we call today symbolic dynamics appeared with the french mathematician Jacques Hadamard in one attempt to simplify one problem, when he used the sequences of symbols to study the distribution of geodesics in certain surfaces. Later in the years 1930-1940, Arnold Hedlund and Marston Morse developed the method in their studies of geodesics of negative curvature, designating these ideas as Symbolic Dynamics.

The fundamental ingredient of a symbolic system is, naturally, the set of symbols $\mathcal{A}$, that we call alphabet. This set $\mathcal{A}$ has a finite number of elements. The most simpler symbolic system that we can create is the fullshift system, that is, the space of all infinite sequences of symbols in $\mathcal{A}$,with

$$\mathcal{A}^{\mathbb{N}} = \{x = (x_i)_{i \in \mathbb{N}} : x_i \in \mathcal{A}\}.$$

This space works as the set of orbits of the dynamical system, and the alphabet $\mathcal{A}$ is its phase space. With the introduction of the shift application $\sigma : \mathcal{A}^{\mathbb{N}} \longmapsto \mathcal{A}^{\mathbb{N}}$, where each element i of the sequence $y = \sigma(x)$ is given by $y_i = x_{i+1}$, we can recuperate usual concepts of dynamical systems as the periodic orbits of period p identified as $\sigma^{k+p}(x) = \sigma^k(x)$, for all k, $n \in \mathbb{N}$. The symbolic system where it cannot be allowed any symbol repetition is a subshift.

If we denote Σ_F as the space of all admissible sequences, then we have $\sigma(\Sigma_F) = \Sigma_F$. In the construction of symbolic spaces Σ_F the family of words allowed can be finite or infinite. When they are finite it receives the name of subshift of finite type.

Given a symbolic system $\Sigma = \Sigma_F$, let $B_m(\Sigma)$ be the set of all words of length m present in the sequences of Σ. The complexity of a symbolic system Σ will be related with the growth rate of the number of elements of $B_m(\Sigma)$ as the length m grows to infinite. This number gives an interpretation of the growing of diversity of sequences, that is, the dynamics of the system.

Definition 1 *Let Σ_F be a symbolic space. We call entropy of Σ_F to the limit $h(\Sigma_F) = \lim_{m\to\infty} \frac{1}{m}\log_2 |B_m(\Sigma_F)|$, where $|B_m(\Sigma_F)|$ represents the number of elements of $B_m(\Sigma_F)$.*

Since $\Sigma_F \subset A^{\mathbb{N}}$, then $|B_m(\Sigma_F)| \le \left|B_m(A^{\mathbb{N}})\right| = |A|^m$ and we have $h(\Sigma_F) \le \log_2 |A|$, the upper limit to entropy.

The most interesting symbolic systems are the ones, from a practical point of view, that allow us without many difficulties to characterize their dynamical aspects. Only in this way the symbolic representation of the system helps to simplify the original problem.

Definition 2 *Let $A = (a_{ij})$, the n-dimension square matrix with elements in the set $\{0, 1\}$. We call Topological Markov Chain (TMC) to the symbolic system Σ_A with elements $x = (x_i)_{i\in\mathbb{N}}$ such that $a_{x_i,x_{i+1}} = 1$.*

Theorem 3 *Let Σ_A be a TMC, with an irreductible matrix A. Then its topological entropy is given by $h(\Sigma_A) = \log \lambda_p(A) = \rho(A)$ where $\lambda_p(A)$ is the Perron eigenvalue of the matrix A, and $\rho(A)$ the spectral radius of A.*

R. Adler, [[1]] made the question "How and to what extent can a dynamical system be represents by a symbolic one?" This question as a very difficult incompatibility to solve: as, usual, the time evolution models are presented in a phase space of differential varieties but any symbolic system is totally disconnected. So, it is impossible to associate this two realities in one-to-one relation, but as said by B.Marcus, [[2]] " A symbolic system will be, always, one good approximation of the model that we want to study." The main question will be always to find the best possible approximation, that is, to define the set of points on the phase space that can be considered equivalent and so represented by an unique symbol. This identification is possible trough the Markov partitions.

2 Codifying the phase space

Codifying the phase space of the dynamical system is the fundamental point to initiate the symbolic representation of it's dynamics. It is necessary to identify the critical points, the discontinuity points and the discontinuity points of the first derivative.

In this article we study the dynamics of the function $f(x) = \frac{x^2-a}{x^2-b}$, with $0 < b < a < 1$. So, let $M = \{x \in \mathbb{R} : x < -\sqrt{b}\}$, $A = \{-\sqrt{b}\}$, $L = \{x \in \mathbb{N} : -\sqrt{b} < x < \sqrt{b}\}$, $C = \{0\}$, $R = \{x \in \mathbb{R} : 0 < x < \sqrt{b}\}$, $B = \{\sqrt{b}\}$, $N = \{x \in \mathbb{R} : \sqrt{b} < x < 1\}$, $U = \{1\}$, $F = \{x \in \mathbb{R} : x > 1\}$. We call $ad(x)$ to the application that establishes the correspondence between $x \in \mathbb{R}$ and the elements of the kneading alphabet $\mathcal{A} = \{M, L, R, N, F, A, C, B, U, \infty\}$. Generically $ad(x) = S_i$, if $x \in S_i$, with S_i the symbols of $\mathcal{A}$. The space of all infinite sequences of symbols of the alphabet $\mathcal{A}$ will be denoted as $\Sigma = \mathcal{A}^{\mathbb{N}}$.

Definition 4 *Let $x \in \mathbb{R}$, and an application f, we define itinerary of x, $it(x)$, or $it_f(x)$ to the sequence of symbols of the kneading alphabet $\mathcal{A}$,*

$$it(x) = ad(x)ad(f(x))ad(f^2(x)) \ldots .$$

We have that $it(x)$ are the elements of Σ.

The relation of order of Σ is $M \prec A \prec L \prec C \prec R \prec B \prec N \prec U \prec F \prec \infty$ induced by the natural order of the real axis.

Definition 5 *We define parity of a sequence* $S_1S_2...S_n$ *as* $\rho(S_1S_2...S_n)$ *and* $\rho(S_1S_2...S_n) = -1$ *if the sum of the symbols* M *and* L *in* $S_1S_2...S_n$ *is odd, and* $\rho(S_1S_2...S_n) = +1$ *if it is even.*

Definition 6 *Let* $P, Q \in \Sigma$, $P_n \neq Q_n$. *We say that* $P \prec Q$ *if*

$$\rho(P_1P_2...P_{n-1}) = +1 \wedge P_n \prec Q_n$$

or

$$\rho(P_1P_2...P_{n-1}) = -1 \wedge Q_n \prec P_n.$$

As it happens in the modal applications of the interval we have: if x,$y \in \mathbb{R}$, such $x \neq y$ then $x < y \Longrightarrow it(x) \preceq it(y)$ and $it(x) \prec it(y) \Longrightarrow x < y$.

Obviously not all elements of Σ are iterations of $x \in \mathbb{R}$ under some application f. To the ones that corresponds to the iterations of some point we will call it admissible sequences.

The relevance that symbolic dynamics achieved in the last two decades is due to the works of Milnor and Thurston, [[5]] and Sousa Ramos. Milnor and Thurston gave us the possibility of the classification of the dynamics through the iterations of some points, called "application kneading sequences". As we can see, for example in [[3]] to $f(x)$ the points that we need to study the dynamics are $x = -\sqrt{b}$, $x = 0$ and $x = \sqrt{b}$. But, since $f^2(-\sqrt{b}) = f^2(\sqrt{b}) = 1$ the knowledge of the itinerary of one point implies the knowledge of the itinerary of the other point, the critical point. So, in symbolic terms the classification of this sequences can be made using a pair of sequences.

Definition 7 *Let* f *an application. We define kneading invariant of* f, $K(f)$ *to the pair* $K(f) = (K_0, K_1)$ *with* $K_0 = it(f(0))$ *and* $K_1 = it(1)$.

Proposition 8 *Let* $f(x) = \frac{x^2-a}{x^2-b}$, *with* $0 < b < a < 1$, *and* $x \in \mathbb{R}$. *We have* $it(f(x)) \preceq K_1$ *or* $K_0 \preceq it(f(x))$, *with* $K(f) = (K_0, K_1)$.

Proof. Let $f(x) = \frac{x^2-a}{x^2-b}$, with $0 < b < a < 1$. If $x \in]-\sqrt{b}, \sqrt{b}[$, we can see that $f(x) \geq f(0)$ and $f(x) \geq 1$. We have by 2 that

$$f(0) \leq f(x) \Longrightarrow it(f(0)) \preceq it(f(x)),$$

but $it(f(0)) = K_0$, so $K_0 \preceq it(f(x))$. If $x \notin]-\sqrt{b}, \sqrt{b}[$, we have $f(x) < 1$, and by 2 we have

$$f(x) < 1 \Longrightarrow it(f(x)) \prec it(1).$$

But, $it(1) = K_1$, so $it(f(x)) \prec K_1$. ■

One of the main contributions of Sousa Ramos in this area was the construction, trough the kneading invariant of the application, of a Markov Matrix. Indeed, for any pair of finite kneading sequences, it is possible to find a Markov partition of the phase space, that is, a finite collection $\mathcal{C} = \{I_0, I_1, ..., I_n\}$ of disjoint open sets, such the closure of its union matches all phase space, and the image by the application of each one of this sets is the union of some elements of $\mathcal{C}$. We have $I_j \frown I_k = \varnothing$, $j \neq k$, and $\bigcup_{j=0}^{n} I_j = \mathbb{R}$ with $f(I_j) = \bigcup_{j_1}^{j_m} I_k$. The Markov transition matrix $A_{K(f)} = (a_{ij})$ associated to the kneading invariant is defined as $a_{ij} = 1$, if $f(I_i) \supset I_j$ and $a_{ij} = 0$ otherwise.

Example 9 *In our function* f*, for* $a \simeq 0,543237$ *and* $b \simeq 0,317003$ *we have* $K(f) = (K_0, K_1) = (FNB, UNMC)$*, a periodic orbit of period* 8*. Making* $x_0 = 0$ *and* $y_0 = \infty$*, with* $x_{n+1} = f(x_n)$ *and* $y_{n+1} = f(y_n)$*, we have the following values ordered in the real axis:* $-\infty < y_3 < -\sqrt{b} < 0 < \sqrt{b} < y_2 < x_2 < 1 < x_1 < \infty$*. Now we can create the finite collection* $\mathcal{C} = \{I_0, I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8\}$ *with* $I_0 =]-\infty, y_3[$*,* $I_1 =]y_3, -\sqrt{b}[$*,* $I_2 =]-\sqrt{b}, 0[$*,* $I_3 =]0, \sqrt{b}[$*,* $I_4 =]\sqrt{b}, y_2[$*,* $I_5 =]y_2, x_2[$*,* $I_6 =]x_2, 1[$*,* $I_7 =]1, x_1[$ *and* $I_8 =]x_1, \infty[$*. Applying* f *we have* $f(I_0) = I_3 \smile I_4 \smile I_5 \smile I_6$*,* $f(I_1) = I_0 \smile I_1 \smile I_2$*,* $f(I_2) = I_8$*,* $f(I_3) = I_8$*,* $f(I_4) = I_0$*,* $f(I_5) = I_1 \smile I_2 \smile I_3$*,* $f(I_6) = I_4$*,* $f(I_7) = I_5$ *and* $f(I_8) = I_6$*. Then the transition matrix of the phase space is*

$$A_{(FNB,UNMC)} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

and using the spectral radius we can calculate its growth number $s \simeq 1,80709$*.*

3 The symbolic way

As is well known, the numerical calculus, specially if we deal with iterations of rational functions, can return a lot of errors hard to control, due to the sensitive iteration of small values under f. So it is important that if we could build the transition matrix using only the symbolic space and its dynamics.

In our function f, we have periodic orbits of $x = 0$ and $x = 1$, that belong to the same orbit, sometimes distinct orbits, or one is periodic and the other pre-periodic, and this ones are our starting point to study the function dynamics. So, we assume that $K(f) = (K_0, K_1)$ is a pair of finite sequences.

Let (x_{ik}) be the sequences given by $x_{0k} = \sigma^k(K_0)$, with $k = 1, 2, ...n_0$ and $x_{1k} = \sigma^k(K_1)$, with $k = 1, 2, ...n_1$. The values n_0 and n_1 are the length of K_0 and K_1, respectively. The points (x_{ik}) will belong to real intervals where the function is increasing or decreasing. We calculate the symbols iteration using the following rules:

(a) if f is increasing	(b) if f is decreasing
$f(]x_{ik}, x_{pq}[=]x_{i,k+1}; x_{p,q+1}[$	$f(]x_{ik}, x_{pq}[=]x_{p,q+1}; x_{i,k+1}[$
$f(]x_{ik}, B[=]x_{i,k+1}; +\infty[$	$f(]-\infty, x_{ik}[=]x_{i,k+1}; U[$
$f(]B, x_{ik}[=]-\infty; x_{i,k+1}[$	$f(]x_{ik}, A[=]-\infty; x_{i,k+1}[$
$f(]x_{ik}, +\infty[=]x_{i,k+1}; U[$	$f(]A, x_{ik}[=]x_{i,k+1}; +\infty[$

Then we build the transition matrix A and can calculate the growth number. To understand better this process lets see the next example.

Example 10 *Let* $K(f) = (FNNMC, UNMMB)$*. So we have* $K_0 = CFNNMC$*,*

$K_1 = \infty UNMMB$ and trough shifting

$$\begin{array}{c|c} x_{01} = \sigma^1(K_0) = FNNMC & x_{11} = \sigma^1(K_1) = UNMMB \\ x_{02} = \sigma^2(K_0) = NNMC & x_{12} = \sigma^2(K_1) = NMMB \\ x_{03} = \sigma^3(K_0) = NMC & x_{13} = \sigma^3(K_1) = MMB \\ x_{04} = \sigma^4(K_0) = MC & x_{14} = \sigma^4(K_1) = MB \\ x_{05} = \sigma^5(K_0) = C & x_{15} = \sigma^5(K_1) = B \end{array} .$$

Comparing the first symbol of each x_{ik} and using the monotonicity of f we can order them on the real axis. We can see that x_{14}, x_{04}, and x_{13} are the lowest values since that f is decreasing in M. Next we look at the second symbol, and we see that $M \prec C \prec B$ in the real axis partition, but before f was decreasing so $B \prec C \prec M$, thus we have $MB \prec MC \prec MMB$. Doing the same to x_{03}, x_{12} and x_{02} we can see that they have N as first symbol and f is increasing, there so, the order will be to the second symbol $M \prec N$ and we have x_{03}, $x_{12} \prec x_{02}$. Now we need to use the third symbol of x_{03} and x_{12}, regarding the aspect that NM has parity -1, so f is decreasing, and the order will be to the third symbol $C \prec M$. In conclusion we have $x_{03} \prec x_{12} \prec x_{02}$ and repeating the process we have the Markov partition

$$x_{14} \prec x_{04} \prec x_{13} \prec A \prec x_{05} \prec x_{15} \prec x_{03} \prec x_{12} \prec x_{02} \prec x_{11} \prec x_{01}$$

and we can build the partition matrix

$$A_{(FNNMC,UNMMB)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} .$$

Until now this calculation is suppressing only the errors in the iteration of the elements, but we need to know K_0 and K_1 by numeric calculations, so its calculation can have also some errors. How to avoid this? Is there a way to calculate K_0 and K_1 without using numerical calculus? The answer is yes! The way is to build a set of rules that allow us to distinguish the admissible sequences from the ones that are not. This way is known as Kneading Sequence Combinatory, and this work intend to contribute to its expansion to the real rational functions, since it is already well know to other types of functions as the polynomial quadratic function.

4 Kneading Sequence Combinatory

As we can evaluate from the graphic of the function

$$f(x) = \frac{x^2 - a}{x^2 - b},$$

with $0 < b < a < 1$, under iteration the symbol U can shift to R or B or N, that is $U \underset{\longrightarrow}{f} R, B, N$. Resuming we have:

$$\begin{array}{l|l}
M \underset{\longrightarrow}{f} M, A, L, C, R, B, N & B \underset{\longrightarrow}{f} \infty \\
A \underset{\longrightarrow}{f} \infty & N \underset{\longrightarrow}{f} M, A, L, C, R, B, N \\
L \underset{\longrightarrow}{f} F & U \underset{\longrightarrow}{f} R, B, N \\
C \underset{\longrightarrow}{f} F & F \underset{\longrightarrow}{f} R, B, N \\
R \underset{\longrightarrow}{f} F & \infty \underset{\longrightarrow}{f} U
\end{array}$$

Since this work still is in an earlier stage of development we will considerate only the sequences of 4 symbols for K_1 and 5 symbols for K_0, at maximum.

We can build an universal rule for all sequences K_0 and K_1, in order to both be admissible.

Proposition 11 *(Rule 1) For $K(f) = (K_0, K_1)$ we have (1) $\sigma^n(K_0) \preceq K_1$ or $K_0 \preceq \sigma^n(K_0)$ and (2) $\sigma^n(K_1) \preceq K_1$ or $K_0 \preceq \sigma^n(K_1)$.*

Proof. This results comes directly from the properties of the function f and its monotonicity. ■

For example $K_1 = FNRFB$ is not admissible since $U \prec \sigma^3(FNRFB) \prec FNRFB$. But this rule alone misses to evaluate a lot of sequences that numerical calculus recognize as admissible. So we need to build more rules, and more refined in order to generate the total compatibility between the numerical calculus and the Kneading Sequences Combinatory.

Theorem 12 *(Rule 2) [João Cabral and Ricardo Severino] Let (K_0, K_1) be a kneading sequence of the real rational functions $f(x) = \frac{x^2-a}{x^2-b}$, with $0 < b < a < 1$. The following chain of inequalities must be satisfied*

$$... \prec \sigma^m(K_0) \prec \sigma^{m-2}(K_1) \prec \sigma^{m-1}(K_0) \prec ...$$

$$... \prec \sigma^2(K_1) \prec \sigma^3(K_0) \prec \sigma(K_1) \prec \sigma^2(K_0) \prec K_1 \prec \sigma(K_0),$$

where the chain stops when the first symbol of one of the sequences $\sigma^m(K_0)$ or $\sigma^m(K_1)$ is not greater than N, for $x > \sqrt{b}$.

Proof. Let $f(x) = \frac{x^2-a}{x^2-b}$, with $0 < b < a < 1$, with the Kneading pair (K_0, K_1). Since $b < a < 1$ we have $f^2(0) < 1 < f(0)$, so $\sigma^2(K_0) \prec K_1 \prec \sigma(K_0)$. For $x > \sqrt{b}$, the function is increasing and from 2 we will have sequentially

$$\begin{array}{l}
\sigma^2(K_0) \prec K_1 \prec \sigma(K_0) \\
\sigma^3(K_0) \prec \sigma(K_1) \prec \sigma^2(K_0) \prec K_1 \prec \sigma(K_0) \\
\sigma^4(K_0) \prec \sigma^2(K_1) \prec \sigma^3(K_0) \prec \sigma(K_1) \prec \sigma^2(K_0) \prec K_1 \prec \sigma(K_0) \\
... \\
\sigma^m(K_0) \prec \sigma^{m-2}(K_1) \prec \sigma^{m-1}(K_0) \prec ... \prec \sigma^2(K_0) \prec K_1 \prec \sigma(K_0)
\end{array} .$$

■

The conditions of Theorem 12 to the initial iterates of $x = f(0)$ and $x = 1$ allow us to characterize the symbolic space where, fixed a kneading sequence, will be the kneading pairs that are formed in it. Comparing this to the what happens in the symbolic space of bimodal applications in the interval, this lines presents a new characteristic: it is possible to determine an upper and lower limit to this lines. Resulting from the Theorem 12 we have the next corollary.

Corollary 13 *Fixing a kneading sequence K_i, the kneading pairs will occupy a region in the symbolic set defined by $\sigma^m(K_i)$ and $\sigma^{m-1}(K_i)$ with m the lowest integer such that the first symbol of the sequence $\sigma^m(K_i)$ is inferior to N.*

Proof. We know that N represent values inferior to one. So, by Theorem 12 we have $\sigma^m(K_0) \prec ... \prec \sigma^{m-1}(K_0)$ and $\sigma^{m-2}(K_1) \prec ... \prec \sigma^{m-3}(K_1)/$ then we have the desired result. ■

Example 14 *Let $K_1 = UNNA$. In this case the chain of symbolic inequalities is*

$$\sigma^4(K_1) \prec \sigma^4(K_0) \prec \sigma^2(K_1) \prec \sigma^3(K_0) \prec \sigma(K_1) \prec \sigma^2(K_0) \prec K_1 \prec \sigma(K_0).$$

We conclude that the kneading sequence K_0 must have the form $K_0 = FNN...$, but, since $\sigma^3(K_1) \prec \sigma^4(K_0) \prec NA$ then the kneading sequences K_0, which can create kneading pairs with $K_1 = UNNA$ are in the symbolic region $FNNA \prec K_0 \prec FNNNA$.

Example 15 *Let $K_0 = FNNMB$. We have $K_1 = UN...$ from theorem 12 and since $\sigma^4(K_0) = MB \prec \sigma^2(K_1) \prec \sigma^3(K_0) = NMB$ the the kneading sequences K_1, which can create kneading pairs with K_0 are in the symbolic region $UNMB \prec K_1 \prec UNNMB$.*

5 Symbolic product for double bifurcation period

One of the great success of the symbolic methods is its capacity to describe the auto-similitarities of the unimodal and bimodal applications on the interval. Indeed, after the work produced by Sousa Ramos, it is today possible to obtain a clear picture of the properties of this similitarities trough a symbolic product of the kneading sequences.

One of the main goals of our research is to see if the entropy is invariant with a $* - product$, as it happens in other already known applications.

If we draw the bifurcation diagram to f, following the curve in the parameter space related with $f^2(1) = 0$, with kneading sequence $K_1 = UNC$ we obtain the figure 1.

With a fast look at the figure 1, we can identify Feigenbaum double period bifurcations.

For $a \simeq 0,47419027$ and $b \simeq 0,23642395$ we have $K_0 = FNNA$, that is, we have a periodic attractive orbit that includes $x = 0$ and $x = 1$, with

$$K(f) = FNNA\infty UNC. \tag{1}$$

For $a \simeq 0,47814508$ and $b \simeq 0,24530767$ we have a similar case with

$$K(f) = FNNMMNNLFNNA\infty UNC. \tag{2}$$

Observing the similarity between this two kneading invariants it is possible to induce that the kneading invariant that will appear after the second double period bifurcation will be, by

$$K(f) = FNNMMNNLFNNLFNNLFNNMMNNLFNNA\infty UNC \tag{3}$$

that indeed exists for $a \simeq 0,47858340$ and $b \simeq 0,24628795$.

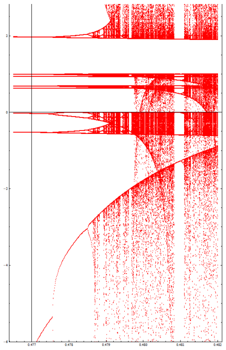

Figure 1: Bifurcation diagram related to the kneading sequence $K_1 = UNC$

Calculating the transition matrix for each one of the kneading sequences 1, 2 and 3 we will have the characteristic polynomials

$$p(t) = (-1+t)t^2(1+t)(1+t^2)(1+t+t^2-t^3),\ p(t) = t^{10}(1+t+t^2-t^3)(-2+t^4)$$

and

$$p(t) = (-1+t)^3t^2(1+t)^3(1+t^2)^3(1+t+t^2-t^3)(1+t^4)^2(1+t^8),$$

respectively. All have spectral radius $s = 1,83929$. So, until now our experimental results show us that identifying the similarities in this real rational functions will share some results already known in the usual unimodal and bimodal applications.

During our experiments we found a big region in the symbolic space that have not matches at all. We resume it in the next proposition.

Proposition 16 *For $K_1 = UNC$, there is no kneading sequence K_0 between $FNNA$ and $FNNMMNLFNNA$.*

Proof. Let's suppose that exists a finite sequence $S = S_1S_2...S_{m-1}X$, with $X = A$ or $X = B$. We show that his sequence violates the necessary conditions for (K_0, UNC) be a kneading invariant.

Let $FNNMMNLFNNA \prec S_1S_2...S_{m-1}X \prec FNNA$. The first four symbols are already determined and so $S = FNNMS_5S_6...S_{m-1}X$. Since the parity of FNN is $+1$ we can write $MMNNL... \prec MS_5S_6...$ and so, $S_5 = M$, $S_6 = N$ and $S_7 = N$. This way we can write $NLF... \prec NS_8S_9$. The alternatives for S_8 are L, R, B or N. But $K_1 = UNC$, so the itinerary of a point in $\sqrt{b} < x < 1$ will necessarily be lower than this one, $NS_8 \prec \sigma(UNC) = NC$. The only possibility to S_8 is L, and $S_9 = N$.

Until now $S = FNNMMNNLFNS_{11}S_{12}...$. By other hand the itinerary of a point $x > 1$ must be bigger that the itinerary of the critical point, so we have $FNNM... \prec FNS_{11}S_{12}...$ and this assure us that $S_{11} = N$ and the symbol S_{12} can be M, A, L, R, B or N. But by theorem 12

$$\sigma^2(FNNM) \prec \sigma^2(FNS_{12}...) \prec \sigma(UNC) \prec \sigma(FNNM...) \prec \sigma(FNS_{12}...)$$

that is $NM... \prec S_{12}S_{13}... \prec NC$.

We conclude that $S_{12} = N$ and $S_{13} = M$, A or L. But, since the parity of the initial subsequence of S is odd than we have $S_{13} \prec A$, and follows that $S_{13} = M$.This is not possible and the sequence should be $S = (FNNMMMNNL)^\infty$, and this contradicts the fact that S should be finite, with a symbol A or B.Then between $FNNA$ and $FNNMMMNLFNNA$ there is no compatible kneading sequence with $K_1 = UNC$.

We can observe that $(FNNMMMNNL)^\infty$ corresponds to the itinerary of the periodic orbit which kneading invariant is $FNNA\infty UNC$. ■

Trough a elaborate and extensive experimental work, and with the help of theorem 12, that would be difficult to write it all in this article, we defined the $* - product$ for the family of real rational functions $f(x) = \frac{x^2-a}{x^2-b}$, with $0 < b < a < 1$, which allow us to describe symbolically the bifurcation phenomena that happens in this functions.

Definition 17 *$* - product$ for $f(x)$.*

Let

$$\overline{A\infty U} = \begin{cases} MMN, \; \rho(P_1...P_{p-1}) = +1 \\ LFN, \; \rho(P_1...P_{p-1}) = -1 \end{cases};$$

$$\overline{B\infty U} = \begin{cases} NMN, \; \rho(P_1...P_{p-1}) = +1 \\ RFN, \; \rho(P_1...P_{p-1}) = -1 \end{cases} \; and \; \overline{C} = \begin{cases} L, \; \rho(Q_2...Q_{q-1}) = +1 \\ R \; \rho(Q_2...Q_{q-1}) = -1 \end{cases}.$$

Fixing K_1, let $f(x)$ be the application such that

$$K(f) = P_1..P_{p-1}X\infty UQ_2...Q_{q-1}C,$$

with $X = A$ or $X = B$. We have the double bifurcation period characterized by:

$$K^*(f) = P_1..P_{p-1}\overline{X\infty U}Q_2...Q_{q-1}\overline{C}P_1..P_{p-1}A\infty UQ_2...Q_{q-1}C.$$

Fixing K_0, let $f(x)$ be the application such that

$$K(f) = UQ_2..Q_{q-1}CP_1...P_{p-1}X\infty,$$

with $X = A$ or $X = B$. We have the double bifurcation period characterized by:

$$K^*(f) = UQ_2..Q_{q-1}\overline{C}P_1...P_{p-1}\overline{X\infty U}Q_2..Q_{q-1}CP_1...P_{p-1}X\infty.$$

The last definition help us to calculate values of entropy that would be impossible to calculate trough numerical calculus, or at least, are really slow regarding our life expectations. Also, it can be used to identify double period bifurcations that sometimes are hidden in the bifurcation diagram, due to numerical calculus approximations errors. We hope this tool would be a useful tool to use in the future research on the dynamics of this family of functions.

References

[1] R.L. Adler. Symbolic dynamics and Markov partitions. Bull.Amer.Math. Soc., 35(1):1-56,1998

[2] B. Marcus. Symbolic dynamics and connections to coding theory, automata theory and system theory. In Proceedings of Symposia in Applied Mathematics, volume 50, pages 95-108. 1995

[3] C. Correia Ramos. Representações, Álgebras C* e Transformações do intervalo. Phd Thesis, Technical University of Lisbon,2006

[4] N. Martins, R.Severino, and J.Sousa Ramos. Isentropic real cubic maps. Internat. J. Bifur. Chaos Appl. Sci. Engr., 13(7):1701-1709,2001.

[5] J. Milnor and W.Thurston. On iterated maps of the interval. In Dynamical Systems proc. Univ. Maryland 1986-87, pages 465-563. Lect. Notes in Math., Springer-Verlag, 1988.